Internal Debt and Welfare

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Appendix

A Derivation of the MNE's maximization problem

The MNE issues new equity E_t in period t with a price q_t on the world capital market. The value of this equity is, thus, $q_t E_t$. Investors purchasing this equity earn next period dividends D_t and capital gains $\dot{q}_t E_t$. They can, however, also invest in other assets in the world market and earn the interest rate r. They are indifferent between investing in the MNE and earning r if

$$\frac{D_t + \dot{q}_t E_t}{q_t E_t} \quad = \quad r. \tag{A.1}$$

Differentiate the value of equity $V_t = q_t E_t$ with respect to time:

$$\dot{V}_t \quad = \quad \dot{q}_t E_t + q_t \dot{E}_t. \tag{A.2}$$

Our objective is to solve (A.2) for the value of equity, V_t. The net profit $\overline{\pi}_t^M$ can either be used to pay dividends D_t or be held as retained earnings RE_t, such that

$$\overline{\pi}_t^M \quad = \quad D_t + RE_t. \tag{A.3}$$

Furthermore, new investment can be financed either through retained earnings RE_t or new equity issues $q_t \dot{E}_t$. Thus, we have

$$(I_t + C(I_t))K_t \quad = \quad RE_t + q_t \dot{E}_t. \tag{A.4}$$

Use Equations (A.3),(A.4), (3) and (5) to solve for $q_t \dot{E}_t$:

$$q_t \dot{E}_t \quad = \quad (I_t + C(I_t))K_t - RE_t = -\pi_t^M + D_t. \tag{A.5}$$

Inserting (A.5) in (A.2), we get

$$\dot{V}_t = \dot{q}_t E_t - \pi_t^M + D_t. \tag{A.6}$$

Next, we solve (A.1) for $\dot{q}_t E_t$ and insert the resulting expression in (A.6) to get

$$\begin{aligned} \dot{V}_t &= r q_t E_t - \pi_t^M \\ &= r V_t - \pi_t^M. \end{aligned} \tag{A.7}$$

Using the integrating factor e^{-rt}, integrating (A.7) from 0 to s, and rearranging, we get

$$V_0 = V_s e^{-rs} + \int_0^s \pi_t^M e^{-rt} dt. \tag{A.8}$$

Finally, evaluating at $s \to \infty$ and using the terminal condition $\lim_{s\to\infty} V_s e^{-rs} = 0$, we get Equation (6).

B Solution of the MNE's optimization problem

Following the assumption of a binding TCR constraint, the optimal internal debt is given by $B_t = bK_t$. Hence, the MNE maximizes Equation (6) over I_t and L_t^m subject to the equation of motion (4) and the initial condition $K(0) = K_0$. Let us use dynamic programming to find the optimum. Define the value function of the maximization problem as $W(K_t)$. The Bellman equation is

$$rW(K_t) = \max_{I_t, L_t^m} \left\{ \pi_t^M + W_K(K_t) I_t K_t \right\}, \tag{B.1}$$

where $W_K(K_t)$ is the derivative of the value function with respect to capital and π_t^M is given by

$$\pi_t^M = F(K_t, L_t^m) - w_t L_t^m - (I_t + C(I_t))K_t - C^B(rb)K_t - \tau[F(K_t, L_t^m) - w_t L_t^m - rbK_t]. \tag{B.2}$$

The first-order conditions are

$$\frac{\partial}{\partial L_t^m} = (1-\tau)(F_L(K_t, L_t) - w_t) = 0, \tag{B.3}$$

$$\frac{\partial}{\partial I_t} = -(1 + C'(I_t))K_t + W_K(K_t)K_t = 0. \tag{B.4}$$

Equation (B.4) gives $W_K(K_t) = 1 + C'(I_t)$. Differentiation of (B.4) with respect to time gives the expression $W_{KK}(K_t)\dot{K}_t = C''\dot{I}_t$. Moreover, using the Envelope theorem, we can differentiate the maximized Bellman equation with respect to capital to get

$$\begin{aligned} rW_K(K_t) &= \pi_K^M + W_K(K_t)I_t + W_{KK}(K_t)I_tK_t \\ &= F_K - (I_t + C(I_t)) - \tau(F_K - rb) - C^B(rb) + W_K(K_t)I_t + W_{KK}(K_t)I_tK_t. \end{aligned} \tag{B.5}$$

Solving (B.5) for $W_{KK}(K_t)$ and inserting the resulting expression in $W_{KK}(K_t)\dot{K}_t = C''\dot{I}_t$, we get

$$C''\dot{I}_t = \frac{\dot{K}_t}{I_tK_t}\left[W_K(K_t)(r - I_t) - (1-\tau)F_K + (I_t + C(I_t)) - \tau rb + C^B(rb)\right]. \tag{B.6}$$

Using Equations (4) and (B.4) to substitute for $\dot{K}_t$ and $W_K(K_t)$ in (B.6) and simplifying, we derive Equation (7). Equation (8) follows directly from (B.3).

C Proof of Lemma 1

To perform the comparative dynamic analysis, I follow Wildasin (2003). Suppose that at time 0, the government permanently increases the tax rate by $d\tau > 0$.

First, derive the impact on the steady state capital stock $\widetilde{K}$, given by $\partial\widetilde{K}/\partial\tau$ (Equation (14)). Differentiate totally Equation (10a) with respect to $\widetilde{K}$ and τ, taking into account that $\widetilde{L}^m = L^m(\widetilde{K})$ according to Equation (11). The resulting expression is

$$\begin{aligned} \left[F_{KK} + F_{KL}\frac{\partial L^m}{\partial\widetilde{K}}\right]\frac{\partial\widetilde{K}}{\partial\tau} &= \frac{-rb(1-\tau) - [r(1-b\tau) + C^B(rb)](-1)}{(1-\tau)^2}, \\ \Leftrightarrow \frac{\partial\widetilde{K}}{\partial\tau} &= \frac{[r(1-b) + C^B(rb)](F_{LL} + G_{LL})}{(1-\tau)^2F_{KK}G_{LL}} = \frac{(F_K - rb)(F_{LL} + G_{LL})}{(1-\tau)F_{KK}G_{LL}}. \end{aligned} \tag{C.1}$$

Equation (C.1) coincides with (14) in Lemma 1.

To derive $\partial K_t/\partial\tau$, differentiate Equations (7) and (4) with respect to $\tau, K_t, \dot{K}_t, I_t$ and $\dot{I}_t$:

$$C''\frac{\partial\dot{I}_t}{\partial\tau} = -(1-\tau)\left(F_{KK} + F_{KL}\frac{\partial L_t^m}{\partial K_t}\right)\frac{\partial K_t}{\partial\tau} + \left[C''(r - I_t) - C'''\dot{I}_t\right]\frac{\partial I_t}{\partial\tau} + (F_K - rb), \tag{C.2}$$

$$\partial\dot{K}_t = I_t\partial K_t + K_t\partial I_t. \tag{C.3}$$

Suppose that the economy is near steady state with $K_t \approx \widetilde{K}, I_t \approx \widetilde{I} = 0, \dot{I}_t \approx 0$. Then, Equation (C.3) becomes

$$\partial I_t = \frac{\partial \dot{K}_t}{\widetilde{K}}. \tag{C.4}$$

Moreover, we can differentiate Equation (C.4) with respect to time, which gives

$$\partial \dot{I}_t = \frac{\partial \ddot{K}_t}{\widetilde{K}}. \tag{C.5}$$

One can now use Equations (C.4), (C.5) and (11) to simplify (C.2):

$$\frac{\partial \ddot{K}_t}{\partial \tau} - r\frac{\partial \dot{K}_t}{\partial \tau} + \frac{(1-\tau)F_{KK}G_{LL}\widetilde{K}}{C''(F_{LL}+G_{LL})}\frac{\partial K_t}{\partial \tau} = \frac{\widetilde{K}(F_K - rb)}{C''}. \tag{C.6}$$

Equation (C.6) is a second-order heterogeneous differential equation in $\partial K_t/\partial\tau$.

The particular solution to (C.6) is found by setting $\partial \ddot{K}_t = \partial \dot{K}_t = 0$. Thus, the particular solution is

$$\frac{\partial K_t}{\partial \tau} = \frac{(F_K - rb)(F_{LL}+G_{LL})}{(1-\tau)F_{KK}G_{LL}}. \tag{C.7}$$

To find the solution to the homogeneous part (i.e., the left-hand side) of (C.6), we suppose that the solution is of the form $\partial K_t/\partial\tau = Ae^{\mu t}$, where A is an undetermined constant. Under the exponential functional form, we have $\partial \dot{K}_t/\partial\tau = \mu\partial K_t/\partial\tau$ and $\partial \ddot{K}_t/\partial\tau = \mu^2\partial K_t/\partial\tau$. Hence, the homogeneous part of (C.6) can be rewritten as

$$\mu^2 - r\mu + \frac{(1-\tau)F_{KK}G_{LL}\widetilde{K}}{C''(F_{LL}+G_{LL})} = 0. \tag{C.8}$$

Equation (C.8) has two solutions for μ, given by

$$\mu_1 = \frac{r - \sqrt{r^2 - \frac{4(1-\tau)F_{KK}G_{LL}\widetilde{K}}{C''(F_{LL}+G_{LL})}}}{2} < 0, \quad \mu_2 = \frac{r + \sqrt{r^2 - \frac{4(1-\tau)F_{KK}G_{LL}\widetilde{K}}{C''(F_{LL}+G_{LL})}}}{2} > 0. \tag{C.9}$$

Therefore, Equation (C.8) has one positive and one negative root. The solution to the homogeneous part is, thus,

$$\frac{\partial K_t}{\partial \tau} = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t}, \tag{C.10}$$

where A_1 and A_2 are undetermined coefficients. The general solution is the sum of the homogeneous and particular solutions:

$$\frac{\partial K_t}{\partial \tau} = \frac{(F_K - rb)(F_{LL} + G_{LL})}{(1-\tau)F_{KK}G_{LL}} + A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t}. \tag{C.11}$$

Invoking the initial condition $\partial K_0/\partial\tau = 0$ and the terminal condition $\lim_{t\to\infty} \partial K_t/\partial\tau = \partial\widetilde{K}/\partial\tau$, one gets $A_1 = -\partial\widetilde{K}/\partial\tau$ and $A_2 = 0$. This completes the proof of Lemma 1. □

D Derivation of the optimal tax rate (Equation (18))

The government's objective function is

$$\int_0^\infty \Omega_t e^{-rt} dt = \int_0^\infty (X_t^W + \beta X_t^E) e^{-rt} dt \tag{D.1}$$

$$= \int_0^\infty \left\{ \tau[F(K_t, L_t^m) - rbK_t] + (1-\tau)w_t L_t^m + G(L_t^d) - (1-\tau)(1-\beta)[G(L_t^d) - w_t L_t^d] \right\} e^{-rt} dt.$$

It maximizes (D.1) subject to $L_t^m = L_t^m(K_t), L_t^d = L_t^d(K_t), w_t = w_t(K_t)$ and Equation (12). The first-order condition is

$$\begin{aligned} \frac{\partial}{\partial\tau} &= \int_0^\infty \Bigg\{ F(K_t, L_t^m) - rbK_t - w_t L_t^m + (1-\beta)[G(L_t^d) - w_t L_t^d] \\ &+ \Bigg[\tau(F_K - rb) + (\tau F_L + (1-\tau)w_t)\frac{\partial L_t^m}{\partial K_t} + (1-\tau)[L_t^m + (1-\beta)L_t^d]\frac{\partial w_t}{\partial K_t} \\ &+ [G_L - (1-\tau)(1-\beta)(G_L - w_t)]\frac{\partial L_t^d}{\partial K_t}\Bigg] \frac{\partial K_t}{\partial\tau} \Bigg\} e^{-rt} dt = 0. \end{aligned} \tag{D.2}$$

Using the labor demand equations $F_L = w$ and $G_L = w$, we can simplify (D.2):

$$\begin{aligned} \frac{\partial}{\partial\tau} &= \int_0^\infty \Bigg\{ F(K_t, L_t^m) - rbK_t - w_t L_t^m + (1-\beta)[G(L_t^d) - w_t L_t^d] \\ &+ \Bigg[\tau(F_K - rb) + (1-\tau)[L_t^m + (1-\beta)L_t^d]\frac{\partial w_t}{\partial K_t} \Bigg] \frac{\partial K_t}{\partial\tau} \Bigg\} e^{-rt} dt = 0. \end{aligned} \tag{D.3}$$

Following Wildasin (2003), I assume that the economy is near its steady state, such that $K_t \approx \widetilde{K}, L_t^m \approx \widetilde{L}^m, L_t^d \approx \widetilde{L}^d, w_t \approx \widetilde{w}$ and $\partial K_t/\partial\tau = \partial\widetilde{K}/\partial\tau(1 - e^{\mu_1 t})$. Thus, (D.3) becomes

$$\int_0^\infty \Big\{ F(\widetilde{K}, \widetilde{L}^m) - rb\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)[G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d] \tag{D.4}$$

$$+ \left[\tau(F_K - rb) + (1-\tau)[\widetilde{L}^m + (1-\beta)\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \widetilde{K}} \right] \frac{\partial \widetilde{K}}{\partial \tau}(1 - e^{\mu_1 t}) \Big\} e^{-rt} dt = 0,$$

where $\partial\widetilde{w}/\partial\widetilde{K}$ is the value of $\partial w_t/\partial K_t$, when evaluated at the steady state. We use (11) and (C.1) to define $\partial\widetilde{w}/\partial\tau$ as

$$\frac{\partial \widetilde{w}}{\partial \tau} \equiv \frac{\partial \widetilde{w}}{\partial \widetilde{K}} \frac{\partial \widetilde{K}}{\partial \tau} = \frac{(F_K - rb)F_{LK}}{(1-\tau)F_{KK}} < 0. \tag{D.5}$$

Integrate the left-hand side of (D.4) to get

$$\begin{aligned} 0 &= F(\widetilde{K}, \widetilde{L}^m) - rb\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)[G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d] \\ &\quad - \frac{\mu_1}{r - \mu_1} \left[\tau(F_K - rb)\frac{\partial \widetilde{K}}{\partial \tau} + (1-\tau)[\widetilde{L}^m + (1-\beta)\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \tau} \right]. \end{aligned} \tag{D.6}$$

We can now use the constant returns property of the production function $F(\cdot)$, which allows $F(\cdot)$ to be represented as $F(K, L^m) = F_K K + F_L L^m = F_K K + wL^m$. Thus, (D.6) becomes

$$\begin{aligned} 0 &= (F_K - rb)\widetilde{K} + (1-\beta)[G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d] \\ &\quad - \frac{\mu_1}{r - \mu_1} \left[\tau(F_K - rb)\frac{\partial \widetilde{K}}{\partial \tau} + (1-\tau)[\widetilde{L}^m + (1-\beta)\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \tau} \right]. \end{aligned} \tag{D.7}$$

Moreover, the partial derivative F_K is homogeneous of degree zero, which means that $0 \cdot F_K = F_{KK}K + F_{KL}L^m$. Therefore, we get

$$(1-\tau)\widetilde{L}^m \frac{\partial \widetilde{w}}{\partial \tau} = \frac{(1-\tau)\widetilde{L}^m (F_K - rb)F_{LK}}{(1-\tau)F_{KK}} = -(F_K - rb)\widetilde{K}. \tag{D.8}$$

Inserting (D.8) in (D.7), denoting the optimal tax as τ^*, and rearranging gives Equation (18).

E Proof of Proposition 1

We use Equations (10a) and (D.6), which determine the steady state capital stock and optimal tax rate, respectively, to derive the effects of a change in b on τ^* and $\widetilde{K}$.

Using (C.1), (D.8) and the labor market clearing condition (9) to simplify (D.6), we can express (10a) and (D.6) as

$$
\begin{aligned}
0 &= F_K(\widetilde{K},\widetilde{L}^m)(1-\tau^*) - r(1-b\tau^*) - C^B(rb), \qquad \text{(E.1)}\\
0 &= F(\widetilde{K},\widetilde{L}^m) - br\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)[G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d]\\
&\quad -\frac{\mu_1}{r-\mu_1}(F_K(\widetilde{K},\widetilde{L}^m) - rb)\left[\tau^*\frac{\partial \widetilde{K}}{\partial \tau} - \frac{\widetilde{K}(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}\right]. \qquad \text{(E.2)}
\end{aligned}
$$

The next step is to totally differentiate (E.1) and (E.2) with respect to $\widetilde{K},\tau^*$, and b. Note, first, that μ_1 depends on τ and $\widetilde{K}$ in the following way:

$$
\begin{aligned}
\frac{d\mu_1}{d\tau} &= \frac{-F_{KK}G_{LL}\widetilde{K}}{C''(F_{LL}+G_{LL})\sqrt{r^2 - \frac{4(1-\tau)F_{KK}G_{LL}\widetilde{K}}{c(F_{LL}+G_{LL})}}} = -\frac{\mu_1(r-\mu_1)}{(r-2\mu_1)(1-\tau)}, \qquad \text{(E.3)}\\
\frac{d\mu_1}{d\widetilde{K}} &= \frac{(1-\tau)F_{KK}G_{LL}(1+\xi)}{C''(F_{LL}+G_{LL})\sqrt{r^2 - \frac{4(1-\tau)F_{KK}G_{LL}\widetilde{K}}{c(F_{LL}+G_{LL})}}} = \frac{\mu_1(r-\mu_1)(1+\xi)}{(r-2\mu_1)\widetilde{K}}, \qquad \text{(E.4)}
\end{aligned}
$$

where

$$
\xi \equiv \widetilde{K}\frac{\left(F_{KKK} + F_{KKL}\frac{\partial \widetilde{L}^m}{\partial \widetilde{K}}\right)G_{LL}(F_{LL}+G_{LL}) - F_{KK}\left[\frac{\partial \widetilde{L}^m}{\partial \widetilde{K}}(G_{LLL}F_{LL} + G_{LL}F_{LLL}) + G_{LL}F_{LLK}\right]}{F_{KK}G_{LL}(F_{LL}+G_{LL})}. \qquad \text{(E.5)}
$$

Using Equations (E.3) (E.4), as well as (11) and (14), the total differential of (E.1) and (E.2) is

$$
\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=J}\begin{pmatrix} d\widetilde{K} \\ d\tau^* \end{pmatrix} = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix} db, \qquad \text{(E.6)}
$$

where

$$
a_{11} \equiv (1-\tau^*)\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}}, \qquad \text{(E.7a)}
$$

$$
a_{12} \equiv -(F_K - rb), \qquad \text{(E.7b)}
$$

$$
\begin{aligned}
a_{21} \equiv \frac{F_K - rb}{r-\mu_1}&\left[\frac{r(1-\tau^*) - \mu_1(1+\tau^*)}{1-\tau^*} - \frac{\mu_1(1-\beta)F_{LL}}{\widetilde{L}^m(F_{LL}+G_{LL})} - \frac{\mu_1\tau^*}{(r-2\mu_1)\widetilde{K}}\frac{\partial \widetilde{K}}{\partial \tau}(r+2\mu_1\xi)\right]\\
&+ \frac{\widetilde{K}(1-\beta\widetilde{L}^d)}{(r-\mu_1)\widetilde{L}^m}\left[r\frac{G_{LL}F_{KK}}{F_{LL}+G_{LL}} + \frac{\mu_1(F_K - rb)}{(r-2\mu_1)\widetilde{K}}(2(r-\mu_1)+r\xi)\right], \qquad \text{(E.7c)}
\end{aligned}
$$

$$a_{22} \equiv -\frac{\mu_1(F_K - rb)\left[\frac{\partial\widetilde{K}}{\partial\tau}(r(1-\tau^*)-2\mu_1)+r\frac{\widetilde{K}(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}\right]}{(r-\mu_1)(r-2\mu_1)(1-\tau^*)}, \tag{E.7d}$$

$$b_1 \equiv r\left(\tau^* - C^{B\,\prime}(rb)\right), \tag{E.7e}$$

$$b_2 \equiv -r\left[\widetilde{K} - \frac{\mu_1}{(r-\mu_1)}\left(2\tau^*\frac{\partial\widetilde{K}}{\partial\tau} - \frac{\widetilde{K}(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}\right)\right]. \tag{E.7f}$$

The determinant of the matrix J is given by

$$\begin{aligned}
|J| &= a_{11}a_{22} - a_{12}a_{21} \\
&= \Bigg\{\frac{\widetilde{K}(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}\left(r(r-3\mu_1)\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}} - \frac{\mu_1(F_K-rb)}{\widetilde{K}}(2(r-\mu_1)+r\xi)\right) \\
&\quad + \frac{(F_K-rb)}{1-\tau^*}\Bigg[(r-2\mu_1)\left(r(1-\tau^*)-\mu_1(2+\tau^*)-\mu_1(1-\tau^*)\frac{(1-\beta)F_{LL}}{\widetilde{L}^m(F_{LL}+G_{LL})}\right) \\
&\quad + \mu_1\tau^*\left(r - \frac{1-\tau^*}{\widetilde{K}}\frac{\partial\widetilde{K}}{\partial\tau}(r+2\mu_1\xi)\right)\Bigg]\Bigg\}\frac{F_K-rb}{(r-\mu_1)(r-2\mu_1)} > 0.
\end{aligned} \tag{E.8}$$

The determinant of J must be positive as required by the second-order condition of the government's maximization problem (i.e., the derivative of (D.6) with respect to τ). Using Cramer's rule, the effects of b on the steady state capital stock and tax rate are:

$$\begin{aligned}
\frac{d\widetilde{K}}{db} &= \frac{1}{|J|}\begin{vmatrix} -b_1 & a_{12} \\ -b_2 & a_{22}\end{vmatrix} = \frac{b_2a_{12}-b_1a_{22}}{|J|} \\
&= \frac{r(F_K-rb)}{|J|(r-2\mu_1)(r-\mu_1)(1-\tau^*)}\Bigg\{\mu_1\tau^*\frac{\partial\widetilde{K}}{\partial\tau}\Bigg[2\mu_1(1-2\tau^*)-r(1-\tau^*) \\
&\quad -\frac{C^{B\,\prime}}{\tau^*}(r(1-\tau^*)-2\mu_1)\Bigg] + \widetilde{K}\Bigg[(r-\mu_1)(r-2\mu_1)(1-\tau^*) \\
&\quad + \mu_1\frac{(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}\left(r-2\mu_1(1-\tau^*)-C^{B\,\prime}r\right)\Bigg]\Bigg\},
\end{aligned} \tag{E.9}$$

$$\begin{aligned}
\frac{d\tau^*}{db} &= \frac{1}{|J|}\begin{vmatrix} a_{11} & -b_1 \\ a_{21} & -b_2\end{vmatrix} = \frac{b_1a_{21}-b_2a_{11}}{|J|} \\
&= \frac{r}{|J|(r-\mu_1)}\Bigg\{(F_K-rb)\tau^*\Bigg[r-\frac{\mu_1(3-\tau^*)}{1-\tau^*}-\mu_1\frac{(1-\beta)F_{LL}}{\widetilde{L}^m(F_{LL}+G_{LL})} \\
&\quad -\frac{\mu_1\tau^*(r+2\mu_1\xi)}{(r-2\mu_1)\widetilde{K}}\frac{\partial\widetilde{K}}{\partial\tau}\Bigg] + (r-\mu_1)(1-\tau^*)\frac{F_{KK}G_{LL}\widetilde{K}}{F_{LL}+G_{LL}} + \frac{\widetilde{K}(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}.
\end{aligned} \tag{E.10}$$

$$\cdot\left[\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}}(\tau^*r+(1-\tau^*)\mu_1)+\frac{\tau^*\mu_1(F_K-rb)}{(r-2\mu_1)\widetilde{K}}(2(r-\mu_1)+r\xi)\right]\Bigg\}-\frac{rC^{B\,\prime}a_{21}}{|J|}.$$

The expression (E.9) contains only negative terms in its first row and both negative and positive terms in the second and third rows. The change in $\widetilde{K}$ has, thus, an ambiguous sign. The same is true for (E.10): while the first row of (E.10) is positive, the second and third rows are either positive or negative.

We are interested in the impact of db, starting from zero internal financing, i.e., $b = 0$. Therefore, we evaluate (E.9) and (E.10) at $b = 0$. Taking into account $C^{B\,\prime}(0) = 0$, we get

$$\frac{d\widetilde{K}}{db}(b=0)=\frac{rF_K}{|J|(r-2\mu_1)(r-\mu_1)(1-\tau^*)}\Bigg\{\mu_1\tau^*\frac{\partial\widetilde{K}}{\partial\tau}\Big[2\mu_1(1-2\tau^*)-r(1-\tau^*)\Big]$$
$$+\widetilde{K}\left[(r-\mu_1)(r-2\mu_1)(1-\tau^*)+\mu_1\frac{(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}\left(r-2\mu_1(1-\tau^*)\right)\right]\Bigg\},\tag{E.11}$$

$$\frac{d\tau^*}{db}(b=0)=\frac{r}{|J|(r-\mu_1)}\Bigg\{F_K\tau^*\Bigg[r-\frac{\mu_1(3-\tau^*)}{1-\tau^*}-\mu_1\frac{(1-\beta)F_{LL}}{\widetilde{L}^m(F_{LL}+G_{LL})}\tag{E.12}$$
$$-\frac{\mu_1\tau^*(r+2\mu_1\xi)}{(r-2\mu_1)\widetilde{K}}\frac{\partial\widetilde{K}}{\partial\tau}\Bigg]+(r-\mu_1)(1-\tau^*)\frac{F_{KK}G_{LL}\widetilde{K}}{F_{LL}+G_{LL}}+\frac{\widetilde{K}(1-\beta\widetilde{L}^d)}{\widetilde{L}^m}\cdot$$
$$\cdot\left[\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}}(\tau^*r+(1-\tau^*)\mu_1)+\frac{\tau^*\mu_1(F_K-rb)}{(r-2\mu_1)\widetilde{K}}(2(r-\mu_1)+r\xi)\right]\Bigg\}.$$

To prove the first part of Proposition 1, take the limit of (E.11) when μ_1 approaches $-\infty$:

$$\lim_{\mu_1\to-\infty}\frac{d\widetilde{K}}{db}(b=0)=\frac{rF_K}{|J|(1-\tau^*)}\left[\frac{\partial\widetilde{K}}{\partial\tau}\left[\tau^*(1-2\tau^*)\right]-\widetilde{K}(1-\beta)\frac{\widetilde{L}^d}{\widetilde{L}^m}(1-\tau^*)\right]<0,$$
$$\text{if}\quad \tau^*<\frac{1}{2},\beta<1.\tag{E.13}$$

The limit of (E.12) when $\mu_1 \to -\infty$ is difficult to sign due to the presence of third derivatives in the term ξ. Therefore, instead of directly evaluating the change in the tax rate, we evaluate it indirectly. Note that the change in the capital stock $\widetilde{K}$ can be split in two effects: a direct effect of b on $\widetilde{K}$ and an indirect effect through the change in the tax rate τ^*:

$$\frac{d\widetilde{K}}{db} \;=\; \frac{\partial\widetilde{K}}{\partial\tau}\frac{d\tau^*}{db}+\frac{\partial\widetilde{K}}{\partial b},\tag{E.14}$$

where the direct effect $\partial\widetilde{K}/\partial b$ is derived by totally differentiating Equation (10a) with respect to $\widetilde{K}$ and b:

$$\frac{\partial\widetilde{K}}{\partial b} = \frac{r\left(C^{B\,\prime}-\tau^*\right)(F_{LL}+G_{LL})}{(1-\tau^*)F_{KK}G_{LL}} > 0. \tag{E.15}$$

The above term is positive because we focus only on internal debt ratios $b<\hat{b}$, where, by definition, $C^{B\,\prime}(r\hat{b})=\tau$. Owing to the convexity of $C^B(\cdot)$, we have $C^{B\prime}(rb)-\tau<0$ for all $b<\hat{b}$. Hence, it is also positive when evaluated at $b=0$. Thus, we can solve for $d\tau^*/db$ from Equation (E.14):

$$\frac{d\tau^*}{db} = \frac{\frac{d\widetilde{K}}{db}-\frac{\partial\widetilde{K}}{\partial b}}{\frac{\partial\widetilde{K}}{\partial\tau}} > 0, \quad \text{if} \quad \mu_1\to-\infty, b=0, \beta<1 \ \text{ and } \ \tau^*<\frac{1}{2}. \tag{E.16}$$

Thus, in a static model, $\tau^*<1/2, \beta<1$ is sufficient for the introduction of internal debt to have a positive effect on the optimal tax rate.

To prove the second part of Proposition 1, evaluate (E.11) and (E.12) at $\beta=1$. Note first that, in this case, the optimal tax rate is determined by

$$\frac{\tau^*}{\widetilde{K}}\frac{\partial\widetilde{K}}{\partial\tau} = \frac{r}{\mu_1}, \tag{E.17}$$

where (E.17) is Equation (18), evaluated at $\beta=1$. Evaluating Equations (E.11) and (E.12) at $\beta=1$ and using (E.17), one gets

$$\frac{d\widetilde{K}}{db}(b=0,\beta=1) = -\frac{r^2F_K\widetilde{K}\mu_1\tau^*}{|J|(r-2\mu_1)(r-\mu_1)(1-\tau^*)} > 0, \tag{E.18}$$

$$\begin{aligned}\frac{d\tau^*}{db}(b=0,\beta=1) = \frac{r}{|J|(r-\mu_1)}\Bigg\{&F_K\tau^*\left[r-\frac{\mu_1(3-\tau^*)}{1-\tau^*}-\frac{r(r+2\mu_1\xi)}{(r-2\mu_1)}\right] \\ &+(r-\mu_1)(1-\tau^*)\frac{F_{KK}G_{LL}\widetilde{K}}{F_{LL}+G_{LL}}+\widetilde{K}\left[\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}}(\tau^*r+(1-\tau^*)\mu_1)\right. \\ &\left.\left.+\frac{\tau^*\mu_1F_K}{(r-2\mu_1)\widetilde{K}}(2(r-\mu_1)+r\xi)\right]\right\}.\end{aligned} \tag{E.19}$$

Equation (E.18) is positive. To sign (E.19), rewrite (E.17) in the case $b=0$ using (C.1):

$$\frac{F_{KK}G_{LL}\widetilde{K}r}{F_{LL}+G_{LL}} = \frac{\tau^*F_K\mu_1}{(1-\tau^*)}. \tag{E.20}$$

Inserting (E.20) in (E.19) and simplifying, one gets

$$\frac{d\tau^*}{db}(b=0,\beta=1) = \frac{rF_K\mu_1\tau^*\left[2\mu_1 - r(2-\tau^*) - (1-\tau^*)r\xi\right]}{|J|(r-\mu_1)(r-2\mu_1)(1-\tau^*)}. \tag{E.21}$$

We now evaluate $|J|$ in the case $\beta = 1, b = 0$:

$$|J|(b=0,\beta=1) = \frac{F_K^2\mu_1}{(r-\mu_1)(r-2\mu_1)(1-\tau^*)}\left[(2-\tau^*)(\mu_1 - r) - (1-\tau^*)r\xi\right] > 0. \tag{E.22}$$

The determinant is positive, and, thus, the second-order condition is satisfied, if $\xi > (2-\tau^*)(\mu_1 - r)/(r(1-\tau^*))$. We use this condition to derive a lower bound for (E.21), which is positive:

$$\frac{d\tau^*}{db}(b=0,\beta=1) > \frac{rF_K\mu_1^2\tau^*}{|J|(r-\mu_1)(r-2\mu_1)(1-\tau^*)} > 0. \tag{E.23}$$

Hence, the change in the optimal tax rate has a positive lower bound and must be positive. Together, Equations (E.18) and (E.23) prove part (b) of Proposition 1 for $\beta = 1$. Since both (E.9) and (E.10) are continuous in β there exist values of β close but not equal to one for which $d\widetilde{K}/db > 0$ and $d\tau^*/db > 0$. Denote the lowest value of β for which these results hold as $\hat{\beta}$. Then, for $\beta \in [\hat{\beta}, 1]$, the steady state capital stock and optimal tax rate are increasing in the TCR.

Lastly, for all values of β not yet considered, i.e., $\beta \in [0, \hat{\beta}[$, the effects of an increase in b are given by (E.9) and (E.10) and are ambiguous. □

F Welfare Effects of Internal Debt

To derive the welfare effects of internal debt use, we begin by expressing the steady state welfare as

$$\widetilde{\Omega} = \tau^*[F(\widetilde{K},\widetilde{L}^m) - rb\widetilde{K}] + (1-\tau^*)\widetilde{w}\widetilde{L}^m + G(\widetilde{L}^d) - (1-\tau^*)(1-\beta)[G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d]. \tag{F.1}$$

First, I prove the following Lemma:

Lemma 2. *Suppose that, starting from $b = 0$, the government allows internal debt by a TCR relaxation $db > 0$ in period 0. If $\mu_1 \to -\infty$, the economy is static and $\Omega_t = \Omega$*

for all $t > 0$. If $\beta < (=)1$, then welfare increases (remains unchanged):

$$\frac{d\Omega}{db}\begin{cases} > 0, & \text{if } \beta < 1 \\ = 0, & \text{if } \beta = 1 \end{cases}. \tag{F.2}$$

Proof: We start by deriving the change in $\widetilde{\Omega}$ by differentiating welfare with respect to b, taking into account the effects of b on τ^* and $\widetilde{K}$. The resulting expression is[1]

$$\begin{aligned}\frac{d\widetilde{\Omega}}{db} &= -r\tau^*\widetilde{K} + \left[F(\widetilde{K},\widetilde{L}^m) - rb\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right]\frac{d\tau^*}{db} \\ &\quad + \left[\tau^*(F_K - rb) + (1-\tau^*)\frac{\partial\widetilde{w}}{\partial\widetilde{K}}(\widetilde{L}^m + (1-\beta)\widetilde{L}^d)\right]\frac{d\widetilde{K}}{db}.\end{aligned} \tag{F.3}$$

Using the government's first-order condition (D.6), we can substitute for the term in brackets in the second row of (F.3). Moreover, we can express $F(\cdot)$ as $F = F_K K + F_L L^m$. Equation (F.3) becomes

$$\frac{d\widetilde{\Omega}}{db} = -r\tau^*\widetilde{K} + \left[(F_K - rb)\widetilde{K} + (1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right]\left[\frac{d\tau^*}{db} + \frac{r-\mu_1}{\mu_1}\frac{\frac{d\widetilde{K}}{db}}{\frac{\partial\widetilde{K}}{\partial\tau}}\right]. \tag{F.4}$$

We can split the effect of b on $\widetilde{K}$ using (E.14) and (E.15). Thus, the welfare change becomes

$$\begin{aligned}\frac{d\widetilde{\Omega}}{db} &= -r\tau^*\widetilde{K} + \left[(F_K - rb)\widetilde{K} + (1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right]\left[\frac{d\tau^*}{db} + \frac{r-\mu_1}{\mu_1}\left(\frac{d\tau^*}{db} + \frac{\frac{\partial\widetilde{K}}{\partial b}}{\frac{\partial\widetilde{K}}{\partial\tau}}\right)\right] \\ &= -r\tau^*\widetilde{K} + \left[(F_K - rb)\widetilde{K} + (1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right]\frac{r}{\mu_1}\left[\frac{d\tau^*}{db} + \frac{(r-\mu_1)(C^{B\prime} - \tau^*)}{(F_K - rb)}\right].\end{aligned} \tag{F.5}$$

Further simplification of (F.5) gives:

$$\begin{aligned}\frac{d\widetilde{\Omega}}{db} &= -\frac{r}{\mu_1}\left\{\widetilde{K}[r(\tau^* - C^{B\prime}) + \mu_1 C^{B\prime}] + \frac{(\tau^* - C^{B\prime})(r-\mu_1)}{F_K - rb}(1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right. \\ &\quad \left. - \left[(F_K - rb)\widetilde{K} + (1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right]\frac{d\tau^*}{db}\right\}.\end{aligned} \tag{F.6}$$

The first row of (F.6) is positive for sufficiently small b, while the second row is negative for $d\tau^*/db > 0$. Hence, the net change in $\widetilde{\Omega}$ is indeterminate.

In the case $\mu_1 \to -\infty$, the welfare effect coincides with the results from Hong and

[1] The derivatives with respect to the labor inputs cancel out.

Smart (2010). To prove this statement, evaluate (F.6) at $\mu_1 \to -\infty$ to get

$$\lim_{\mu_1 \to -\infty} \frac{d\widetilde{\Omega}}{db} = r\left[\frac{(\tau^* - C^{B\prime})(1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)}{F_K - rb} - C^{B\prime}\widetilde{K}\right] > 0, \quad \text{if } b = 0, \beta < 1. \tag{F.7}$$

Evaluated at $b = 0, \beta < 1$, the whole expression is positive because of $C^{B\prime}(0) = 0$.

Moreover, in the case $\mu_1 \to -\infty$, the economy is static and $d\Omega_t = d\widetilde{\Omega} \equiv d\Omega$ for all $t > 0$, while period 0 with the initial condition $dK_0 = 0$ cannot be observed. Thus, (F.7) holds for all periods $t > 0$. □

Now, I proceed with the proof of Proposition 2. Evaluate (F.6) at $\beta = 1$:

$$\frac{d\widetilde{\Omega}}{db}(\beta = 1) = -\frac{r\widetilde{K}}{\mu_1}\left[r(\tau^* - C^{B\prime}) + \mu_1 C^{B\prime} - (F_K - rb)\frac{d\tau^*}{db}(\beta = 1)\right]. \tag{F.8}$$

Use Equations (C.1) and (E.15) to derive the expression

$$\frac{\partial \widetilde{K}}{\partial b} = \frac{r(C^{B\prime} - \tau^*)}{(F_K - rb)}\frac{\partial \widetilde{K}}{\partial \tau}. \tag{F.9}$$

Equations (F.9) and (E.14) together give

$$\frac{d\widetilde{K}}{db} = \frac{\partial \widetilde{K}}{\partial \tau}\frac{d\tau^*}{db} + \frac{\partial \widetilde{K}}{\partial b} = \frac{\partial \widetilde{K}}{\partial \tau}\left[\frac{d\tau^*}{db} + \frac{r(C^{B\prime} - \tau^*)}{F_K - rb}\right]. \tag{F.10}$$

To simplify (F.8), evaluate (F.10) at $\beta = 1$ and insert it in (F.8):

$$\frac{d\widetilde{\Omega}}{db}(\beta = 1) = r\widetilde{K}\left[\frac{(F_K - rb)\frac{d\widetilde{K}}{db}(\beta = 1)}{\mu_1 \frac{\partial \widetilde{K}}{\partial \tau}} - C^{B\prime}\right] > 0, \quad \text{if } b = 0. \tag{F.11}$$

Evaluated at $b = 0$, the first term in brackets in (F.11) is unambiguously positive, while the second term is zero. Thus, (F.11) is positive at $b = 0$, which proves Equation (22) from Proposition 2. Since welfare is continuous in β, this result holds also for values of β sufficiently close but not equal to one. Define the lowest value of β that satisfies (F.11) as $\underline{\beta}^1$. Then, (F.11) is satisfied for $\beta \in [\underline{\beta}^1, 1]$. To derive the short term welfare change, define short term welfare Ω_0 as

$$\begin{aligned} \Omega_0 = {} & \tau^*[F(K_0, L_0^m) - rbK_0] + (1 - \tau^*)w_0 L_0^m + G(L_0^d) \\ & - (1 - \tau^*)(1 - \beta)[G(L_0^d) - w_0 L_0^d], \end{aligned} \tag{F.12}$$

where a subscript 0 denotes the initial period where the economy is in steady state (prior to the disturbance) such that $K_0 = \widetilde{K}, L_0^m = \widetilde{L}^m, w_0 = \widetilde{w}, L_0^d = \widetilde{L}^d$. The capital stock cannot change in time period zero, $dK_0/db = 0$, as it is a stock variable. Consequently, the wage rate and the labor demands also remain unchanged at time period 0. Hence, the initial impact on welfare of a change in internal debt in period 0 is

$$\frac{d\Omega_0}{db} = -\tau^* r\widetilde{K} + \left[F(\widetilde{K},\widetilde{L}^m) - rb\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right]\frac{d\tau^*}{db}. \quad \text{(F.13)}$$

Evaluate (F.13) at $\beta = 1$ using the the constant returns property $F = F_K K + F_L L^m$:

$$\frac{d\Omega_0}{db}(\beta = 1) = -\tau^* r\widetilde{K} + \left[F_K(\widetilde{K},\widetilde{L}^m) - rb\right]\widetilde{K}\frac{d\tau^*}{db}(\beta = 1). \quad \text{(F.14)}$$

We add and subtract $rC^{B\prime}\widetilde{K}$ from (F.14), and use (F.10) to simplify the resulting expression:

$$\begin{aligned}
\frac{d\Omega_0}{db}(\beta = 1) &= -\tau^* r\widetilde{K} - rC^{B\prime}\widetilde{K} + rC^{B\prime}\widetilde{K} + (F_K - rb)\,\widetilde{K}\frac{d\tau^*}{db}(\beta = 1) \\
&= \widetilde{K}\left[r(C^{B\prime} - \tau^*) + (F_K - rb)\frac{d\tau^*}{db}(\beta = 1) - rC^{B\prime}\right] \\
&= \widetilde{K}\left[\frac{(F_K - rb)\frac{d\widetilde{K}}{db}(\beta = 1)}{\frac{\partial\widetilde{K}}{\partial\tau}} - rC^{B\prime}\right] < 0, \quad \text{if } b = 0. \quad \text{(F.15)}
\end{aligned}$$

Equation (F.15) proves Equation (21) from Proposition 2. Following the same intuition as before, there exists some $\underline{\beta}^2 < 1$ such that (F.15) holds for $\beta \in [\underline{\beta}^2, 1]$. Equations (F.11) and (F.15) are simultaneously fulfilled for $\beta \in [\underline{\beta}, 1]$, where $\underline{\beta} = \max\{\underline{\beta}^1, \underline{\beta}^2\}$.

Lastly, both (F.6) and (F.13) are ambiguous for $\beta < \underline{\beta}$ and $\mu_1 \in]-\infty, 0[$. □

G Proof of Proposition 3

The government's maximization problem is

$$\max_{\tau,b} \int_0^\infty \Omega_t e^{-rt} dt. \quad \text{(G.1)}$$

The optimal tax rate is given by Equation (18), where b is replaced by its optimal value b^*. The first-order condition with respect to b is

$$\frac{\partial}{\partial b} = \int_0^\infty \left\{ -\tau r K_t + \left[\tau(F_K - rb) + (1-\tau)[L_t^m + (1-\beta)L_t^d]\frac{\partial w_t}{\partial K_t}\right]\frac{\partial K_t}{\partial b}\right\} e^{-rt} dt = 0,$$

(G.2)

where $\partial K_t/\partial b = (1-e^{\mu_1 t})\partial\widetilde{K}/\partial b$ and $\partial\widetilde{K}/\partial b$ is given by (E.15). We first prove that $0 \leq b^* < \hat{b}$. First, rewrite (E.15) as

$$\frac{\partial\widetilde{K}}{\partial b} = \frac{r(C^{B\prime}-\tau)(F_{LL}+G_{LL})}{(1-\tau)F_{KK}G_{LL}} = \frac{r(C^{B\prime}-\tau)}{F_K - rb}\frac{\partial\widetilde{K}}{\partial\tau} > 0. \tag{G.3}$$

Moreover, the first-order condition with respect to τ is

$$\begin{aligned}&\int_0^\infty \left[(F_K - rb)K_t + (1-\beta)[G(L_t^d) - w_t L_t^d]\right] e^{-rt}dt \\ &= -\int_0^\infty \left[\tau(F_K - rb) + (1-\tau)[L_t^m + (1-\beta)L_t^d]\frac{\partial w_t}{\partial K_t}\right]\frac{\partial K_t}{\partial\tau}e^{-rt}dt. \end{aligned}\tag{G.4}$$

Using (G.3) and (G.4) to evaluate (G.2), we get

$$\begin{aligned}\frac{\partial}{\partial b} &= \int_0^\infty \left\{ -\tau r K_t + \frac{r(C^{B\prime}-\tau)\left[\tau(F_K - rb) + (1-\tau)[L_t^m + (1-\beta)L_t^d]\frac{\partial w_t}{\partial K_t}\right]}{F_K - rb}\frac{\partial K_t}{\partial\tau}\right\} e^{-rt}dt \\ &= \int_0^\infty \left\{-\tau r K_t - r(C^{B\prime}-\tau)\frac{(F_K - rb)K_t + (1-\beta)[G(L_t^d) - w_t L_t^d]}{F_K - rb}\right\} e^{-rt}dt \\ &= r\int_0^\infty \left\{-C^{B\prime}K_t + (\tau - C^{B\prime})(1-\beta)\frac{G(L_t^d) - w_t L_t^d}{F_K - rb}\right\} e^{-rt}dt. \end{aligned}\tag{G.5}$$

Evaluated at $b = 0$, (G.5) is strictly positive for $\beta < 1$ and zero for $\beta = 1$ (owing to $C^{B\prime}(0) = 0$). Thus, b^* is strictly positive for $\beta < 1$. Moreover, at $\beta = 1$, (G.5) is strictly negative for $b > 0$. Hence, $b^* = 0$ for $\beta = 1$. Next, evaluate (G.2) at $b = \hat{b}$. In this case, $C^{B\prime}(\hat{b}) = \tau$ and $\partial\widetilde{K}/\partial b = 0$. Hence, the first-order condition (G.2) becomes

$$\frac{\partial}{\partial b}(b = \hat{b}) = -\int_0^\infty \tau r K_t e^{-rt}dt < 0. \tag{G.6}$$

Hence, $b^* < \hat{b}$. This proves the first part of Proposition 3. To prove the second part, evaluate the integrand of (G.2) at $t = 0$, taking into account $\partial K_0/\partial b = 0$:

$$\frac{\partial\Omega_0}{\partial b} = -\tau r K_0 < 0. \tag{G.7}$$

Therefore, there is a negative welfare effect in period 0. Because the change in welfare $\partial\Omega_t/\partial b$ is continuous in time, it is also negative for slightly positive values of t. Denote the largest value of t for which $\partial \Omega_t / \partial b$ is negative as t^*. Then, we must have

$$\int_0^{t^*} \frac{\partial \Omega_t}{\partial b} e^{-rt} < 0. \tag{G.8}$$

□

H Properties of the function $I_t(K_t, \tau_t)$

This section derives the first and second partial derivatives of the function $I_t(K_t, \tau_t)$. Start with $\partial I_t / \partial K_t$. First, we take the total differential of (B.4) with respect to I_t and K_t, taking into account that $C'' = c$, and rearrange to get

$$\frac{\partial I_t}{\partial K_t} = \frac{W_{KK}(K_t)}{c}. \tag{H.1}$$

It remains to derive W_{KK}. To do so, differentiate Equation (B.5) with respect to K_t, taking into account that I_t and L_t^m depend on K_t:

$$rW_{KK} = \pi^M_{KK} + 2W_{KK}I_t + W_{KKK}I_tK_t + \left[\pi^M_{KI} + W_K + W_{KK}K\right]\frac{\partial I_t}{\partial K_t}, \tag{H.2}$$

where

$$\pi^M_{KK} = (1-\tau_t)\left(F_{KK} + F_{KL}\frac{\partial L^m}{\partial K}\right) = (1-\tau_t)\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}}, \tag{H.3a}$$

$$\pi^M_{KI} = -(1 + cI_t). \tag{H.3b}$$

Using Equation (B.4) to express W_K, as well as (H.1), (H.3a) and (H.3b), Equation (H.2) becomes

$$rW_{KK} = (1-\tau_t)\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}} + 2W_{KK}I_t + W_{KKK}I_tK_t + W_{KK}K\frac{W_{KK}}{c}. \tag{H.4}$$

Equation (H.4) is quadratic in W_{KK}. To solve it, rewrite it first as

$$W^2_{KK} + \beta_t W_{KK} + \gamma_t = 0, \tag{H.5}$$

where

$$\beta_t \equiv \frac{c(2I_t - r)}{K_t}, \tag{H.6}$$

$$\gamma_t \equiv \frac{c}{K_t}\left[\frac{(1-\tau_t)F_{KK}G_{LL}}{F_{LL}+G_{LL}} + W_{KKK}I_tK_t\right]. \quad \text{(H.7)}$$

Now evaluate (H.5) around steady state, where $I_t \approx 0, K_t \approx \widetilde{K}$ and $\tau_t \approx \widetilde{\tau}$, and denote the corresponding parameters as $\widetilde{\beta}$ and $\widetilde{\gamma}$. We get

$$\widetilde{\beta} = -\frac{rc}{\widetilde{K}} < 0, \quad \text{(H.8)}$$

$$\widetilde{\gamma} = \frac{(1-\widetilde{\tau})cF_{KK}G_{LL}}{\widetilde{K}(F_{LL}+G_{LL})} < 0. \quad \text{(H.9)}$$

The two solutions to (H.5) are given by

$$W_{KK} = \frac{c}{2\widetilde{K}}\left[r - \sqrt{r^2 - \frac{4(1-\widetilde{\tau})F_{KK}G_{LL}\widetilde{K}}{c(F_{LL}+G_{LL})}}\right] = \frac{c\mu_1}{\widetilde{K}} < 0, \quad \text{(H.10)}$$

$$W_{KK} = \frac{c}{2\widetilde{K}}\left[r + \sqrt{r^2 - \frac{4(1-\widetilde{\tau})F_{KK}G_{LL}\widetilde{K}}{c(F_{LL}+G_{LL})}}\right] = \frac{c\mu_2}{\widetilde{K}} > 0, \quad \text{(H.11)}$$

where μ_1 and μ_2 are defined in Equation (C.9). Note that the value function $W(K)$ must be concave in the capital stock when the objective function as well as the rate of change of the capital stock I_tK_t are concave in K and I. Since this is satisfied, the solution to W_{KK} is given by the negative root, (H.10). Thus, (H.1) and (H.10) together give

$$\frac{\partial I_t}{\partial K_t} = \frac{\mu_1}{\widetilde{K}} < 0. \quad \text{(H.12)}$$

Next, we derive the second derivative of investment with respect to capital. Differentiation of (H.1) with respect to K_t gives

$$\frac{\partial^2 I_t}{\partial K_t^2} = \frac{W_{KKK}(K_t)}{c}. \quad \text{(H.13)}$$

It remains to derive W_{KKK}. I follow Kimball (2014). To ease notation, define the right-hand side of (B.1) as H_t, i.e., $H_t \equiv \pi_t^M + W_K(K_t)I_tK_t$.[2] Now, differentiate the Bellman equation $rW(K_t) = H_t$ with respect to capital without invoking the Envelope theorem:

$$rW_K = H_K + H_I\frac{\partial I_t}{\partial K_t}, \quad \text{(H.14)}$$

[2]Kimball (2014) proposes the term “prevalue function” for H_t, as it is the maximization of H_t that yields the value function.

where

$$H_K = \pi_K^M + W_K I_t + W_{KK} I_t K_t, \tag{H.15a}$$
$$H_I = -(1 + cI_t)K_t + W_K K_t = 0, \tag{H.15b}$$

where $H_I = 0$ owing to the first-order condition (B.4). Next, differentiate (H.14) with respect to capital:

$$rW_{KK} = H_{KK} + 2H_{KI}\frac{\partial I_t}{\partial K_t} + H_{II}\left(\frac{\partial I_t}{\partial K_t}\right)^2 + H_I\frac{\partial^2 I_t}{\partial K_t^2}, \tag{H.16}$$

where

$$H_{KK} = \pi_{KK}^M + 2W_{KK} I_t + W_{KKK} I_t K_t, \tag{H.17a}$$
$$H_{KI} = \pi_{KI}^M + W_K + W_{KK} K_t, \tag{H.17b}$$
$$H_{II} = -cK_t, \tag{H.17c}$$

and π_{KK}^M, π_{KI}^M are defined in (H.3a), (H.3b). Lastly, differentiate (H.16) with respect to K_t:

$$rW_{KKK} = H_{KKK} + 3H_{KKI}\frac{\partial I_t}{\partial K_t} + 3H_{IIK}\left(\frac{\partial I_t}{\partial K_t}\right)^2 + 3\frac{\partial^2 I_t}{\partial K_t^2}\left[H_{KI} + H_{II}\frac{\partial I_t}{\partial K_t}\right] + H_I\frac{\partial^3 I_t}{\partial K_t^3}, \tag{H.18}$$

where

$$H_{KKK} = \pi_{KKK}^M + 3W_{KKK} I_t + W_{KKKK} I_t K_t, \tag{H.19a}$$
$$H_{KKI} = 2W_{KK} + W_{KKK} K_t, \tag{H.19b}$$
$$H_{IIK} = -c, \tag{H.19c}$$
$$\pi_{KKK}^M = (1-\tau_t)\frac{F_{KK} G_{LL}}{F_{LL} + G_{LL}}\frac{\xi_t}{K_t}. \tag{H.19d}$$

The term ξ_t is the value of ξ, defined in Equation (E.5), evaluated in period t. Moreover, according to (B.4), $H_I = 0$. Additionally, Equation (H.1) gives

$$H_{KI} + H_{II}\frac{\partial I_t}{\partial K_t} = 0.$$

Thus, (H.18) becomes

$$rW_{KKK} = H_{KKK} + 3H_{KKI}\frac{\partial I_t}{\partial K_t} + 3H_{IIK}\left(\frac{\partial I_t}{\partial K_t}\right)^2. \tag{H.20}$$

Now, evaluate (H.20) around steady state, where $I_t \approx 0, K_t \approx \widetilde{K}, \tau_t \approx \widetilde{\tau}$, and using Equations(H.12), (H.10), (H.19a)-(H.19d). The resulting expression is

$$W_{KKK} = \frac{1}{r - 3\mu_1}\left[\frac{(1-\widetilde{\tau})F_{KK}G_{LL}\xi}{\widetilde{K}(F_{LL}+G_{LL})} + 3c\left(\frac{\mu_1}{\widetilde{K}}\right)^2\right]. \tag{H.21}$$

One way to simplify (H.21) is to solve (C.8) for the term containing the second derivatives of the production function. Inserting the resulting expression in (H.21), one gets

$$W_{KKK} = \frac{\mu_1 c}{(r-3\mu_1)\widetilde{K}^2}\left[(r-\mu_1)\xi + 3\mu_1\right]. \tag{H.22}$$

Thus, (H.13) and (H.22) together give

$$\frac{\partial^2 I_t}{\partial K_t^2} = \frac{\mu_1}{(r-3\mu_1)\widetilde{K}^2}\left[(r-\mu_1)\xi + 3\mu_1\right]. \tag{H.23}$$

Next, use (H.1) to derive the cross-derivative

$$\frac{\partial^2 I_t}{\partial K_t \partial \tau_t} = \frac{1}{c}\frac{\partial W_{KK}}{\partial \tau_t}, \tag{H.24}$$

where the effect of the statutory tax rate on the second derivative of the value function is determined by (H.5). A total differential of (H.5) gives

$$\frac{\partial W_{KK}}{\partial \tau_t} = -\frac{W_{KK}\frac{\partial \beta_t}{\partial \tau_t} + \frac{\partial \gamma_t}{\partial \tau_t}}{2W_{KK} + \beta_t}. \tag{H.25}$$

Using Equations (H.6) and (H.7), one can derive

$$\frac{\partial \beta_t}{\partial \tau_t} = 2\frac{c}{K_t}\frac{\partial I_t}{\partial \tau_t}, \tag{H.26}$$

$$\frac{\partial \gamma_t}{\partial \tau_t} = \frac{c}{K_t}\left[-\frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}} + W_{KKK}K_t\frac{\partial I_t}{\partial \tau_t}\right]. \tag{H.27}$$

In deriving the above expressions, we take into account that the capital stock K_t does not react to changes in τ_t, as it is a stock and cannot change immediately. Using (H.24)-(H.27), as well as (H.8) and (H.10), we get

$$\frac{\partial^2 I_t}{\partial K_t \partial \tau_t} = \frac{1}{(r-2\mu_1)c}\left[\frac{\partial I_t}{\partial \tau_t}\frac{\mu_1 c}{\widetilde{K}}\left(2 + \frac{(r-\mu_1)\xi + 3\mu_1}{(r-3\mu_1)\widetilde{K}}\right) - \frac{F_{KK}G_{LL}}{F_{LL}+G_{LL}}\right]. \tag{H.28}$$

It remains to derive the first and second derivatives of investment with respect to the tax rate. To do so, differentiate totally (B.4) with respect to I_t and W_K to get

$$\partial I_t = \frac{1}{c}\partial W_K. \tag{H.29}$$

Then, rewrite (B.5) in the case of a time-varying tax rate as

$$(r - I_t)W_K = F_K - \tau_t(F_K - rb) - (I_t + C(I_t)) - C^B(rb) + W_{KK}I_tK_t. \tag{H.30}$$

Then, differentiate (H.30) with respect to I_t, W_K, W_{KK} and τ_t:

$$(r - I_t)\partial W_K = -(F_K - rb)\partial\tau_t - (1 + cI_t - W_K - W_{KK}K_t)\partial I_t + I_tK_t\partial W_{KK}. \tag{H.31}$$

We can use (H.29) and (H.31) to derive

$$\frac{\partial I_t}{\partial \tau_t} = \frac{-(F_K - rb) + I_tK_t\frac{\partial W_{KK}}{\partial \tau_t}}{rc + 1 - W_K - W_{KK}K_t}. \tag{H.32}$$

Around steady state, $I_t \approx 0, W_K = 1 + c \cdot 0, W_{KK} = \mu_1 c/\widetilde{K}$. Thus, (H.32) simplifies to

$$\begin{aligned}\frac{\partial I_t}{\partial \tau_t} &= -\frac{(F_K - rb)}{(r - \mu_1)c} \\ &= -\frac{\mu_1(F_K - rb)(F_{LL} + G_{LL})}{(1 - \widetilde{\tau})F_{KK}G_{LL}\widetilde{K}} = -\frac{\mu_1}{\widetilde{K}}\frac{\partial \widetilde{K}}{\partial \tau} < 0,\end{aligned} \tag{H.33}$$

where the last row was derived by the use of (C.8) to substitute for $(r - \mu_1)c$. Lastly, derive the second derivative of investment with respect to the tax rate, by using (H.32). It is given by

$$\frac{\partial^2 I_t}{\partial \tau_t^2} = \frac{\frac{\partial I_t}{\partial \tau_t}\left[2K_t\frac{\partial W_{KK}}{\partial \tau_t} + \frac{\partial W_K}{\partial \tau_t}\right]}{rc + 1 - W_K - W_{KK}K_t}. \tag{H.34}$$

Using Equations (H.25), (H.26), (H.27) and (H.33), the above equation can be expressed, around a steady state, as

$$\begin{aligned}\frac{\partial^2 I_t}{\partial \tau_t^2} &= -\frac{(F_K - rb)}{(r - \mu_1)^2(r - 2\mu_1)c} \times \\ &\quad \times \left[\frac{\partial I_t}{\partial \tau_t}\left(r + 2\mu_1 + 2\mu_1\frac{(r - \mu_1)\xi + 3\mu_1}{(r - 3\mu_1)}\right) - \frac{2\widetilde{K}F_{KK}G_{LL}}{c(F_{LL} + G_{LL})}\right].\end{aligned} \tag{H.35}$$

I Proof of Proposition 4

To derive the optimal tax rate, note first that period t welfare is given by

$$\begin{aligned}\Omega_t &= \tau_t[F(K_t, L_t^m(K_t)) - rbK_t] + (1-\tau_t)w_t(K_t)L_t^m(K_t) + G(L_t^d(K_t)) \\ &\quad -(1-\tau_t)(1-\beta)[G(L_t^d(K_t)) - w_t(K_t)L_t^d(K_t)]. \end{aligned} \tag{I.1}$$

Denote the government's value function as $U(K_t)$. Then, its maximization problem can be written as

$$rU(K_t) = \max_{\tau_t} \{\Omega_t + U_K(K_t)I_tK_t\}. \tag{I.2}$$

To simplify the exposition of the proof, use the notation $A(K_t, \tau_t) \equiv I_tK_t$ and $H^G \equiv \Omega_t + U_K(K_t)A(K_t, \tau_t)$, where H^G can be referred to as the prevalue function of the government. Since Ω_t in linear in the tax rate τ_t, we need to assume that $A(K_t, \tau_t)$ is strictly concave in the tax rate for the maximization problem to be well-behaved. Thus, we require $A_{\tau\tau} = (\partial^2 I_t/\partial\tau_t^2)K_t < 0$, where $\partial^2 I_t/\partial\tau_t^2$ is given by (H.35). The first-order condition of the government is given by

$$\begin{aligned}H_\tau^G &= F(K_t, L_t^m(K_t)) - rbK_t - w_t(K_t)L_t^m(K_t) + (1-\beta)[G(L_t^d(K_t)) - w_t(K_t)L_t^d(K_t)] \\ &\quad +U_K(K_t)A_\tau = 0, \end{aligned} \tag{I.3}$$

where $A_\tau = (\partial I_t/\partial\tau_t)K_t$. We derive first the steady state tax rate $\widetilde{\tau}$. To find it, one needs first the value of $U_K(K_t)$, evaluated in steady state. It is found by a differentiation of the maximized Bellman equation with respect to capital:

$$rU_K = \Omega_K + U_{KK}A + U_KA_K, \tag{I.4}$$

where

$$\begin{aligned}\Omega_K &= \tau_t\left(F_K - rb + F_L\frac{\partial L_t^m}{\partial K_t}\right) + (1-\tau_t)w_t\frac{\partial L_t^m}{\partial K_t} + G_L\frac{\partial L_t^d}{\partial K_t} \\ &\quad -(1-\beta)(1-\tau_t)(G_L - w_t)\frac{\partial L_t^d}{\partial K_t} + \frac{\partial w_t}{\partial K_t}(1-\tau_t)(L_t^m + (1-\beta)L_t^d) \\ &= \tau_t(F_K - rb) + \frac{\partial w_t}{\partial K_t}(1-\tau_t)(L_t^m + (1-\beta)L_t^d), \end{aligned} \tag{I.5}$$

$$A_K = I_t + \frac{\partial I_t}{\partial K_t}K_t. \tag{I.6}$$

Around steady state, $K_t \approx \widetilde{K}, I_t \approx 0$ and $\partial I_t/\partial K_t$ is given by (H.12). Thus, one can solve (I.4) for U_K:

$$U_K(\widetilde{K}) = \frac{\widetilde{\tau}(F_K - rb) + (1-\widetilde{\tau})\frac{\partial \widetilde{w}}{\partial K}(\widetilde{L}^m + (1-\beta)\widetilde{L}^d)}{r - \mu_1}. \tag{I.7}$$

Moreover, one can evaluate (I.3) around steady state, using (H.33) and (I.7). The resulting expression is (18), where one substitutes τ^* with $\widetilde{\tau}$. Hence, $\widetilde{\tau}$ is equal to τ^* from Section 3.

To derive the path of the optimal tax rate, totally differentiate (I.3) with respect to time, taking into account that both τ_t and K_t are functions of time. The resulting expression is

$$H^G_{\tau\tau}\dot{\tau}_t + H^G_{\tau K}\dot{K}_t = 0, \tag{I.8}$$

where

$$\begin{aligned}
H^G_{\tau\tau} &= U_K A_{\tau\tau} < 0, & \text{(I.9)}\\
H^G_{\tau K} &= \Omega_{\tau K} + U_{KK}A_\tau + U_K A_{\tau K}, & \text{(I.10)}\\
\Omega_{\tau K} &= F_K - rb + F_L\frac{\partial L^m_t}{\partial K_t} - w_t\frac{\partial L^m_t}{\partial K_t} + (1-\beta)(G_L - w_t)\frac{\partial L^d_t}{\partial K_t} - \frac{\partial w_t}{\partial K_t}(L^m_t + (1-\beta)L^d_t) & \\
&= F_K - rb - \frac{\partial w_t}{\partial K_t}(L^m_t + (1-\beta)L^d_t), & \text{(I.11)}\\
A_{\tau K} &= \frac{\partial I_t}{\partial \tau_t} + \frac{\partial^2 I_t}{\partial \tau_t \partial K_t}K_t. & \text{(I.12)}
\end{aligned}$$

The expression (I.9) is negative due to our assumption about the concavity of the objective function. One can solve (I.8) for τ_t. Near steady state $\dot{K}_t$ can be approximated (using (4) and (H.12)) as

$$\begin{aligned}
\dot{K}_t = I_t(K_t)K_t &\approx \left[\frac{\partial I_t}{\partial K_t}K_t + I_t\right](K_t - \widetilde{K})\\
&\approx \mu_1(K_t - \widetilde{K}),
\end{aligned} \tag{I.13}$$

where in the second row of (I.13) we used the steady state conditions $K_t \approx \widetilde{K}, I_t \approx \widetilde{I} = 0$ and Equation (H.12). Starting from an initial capital stock K_0, the solution to the differential equation (I.13) is

$$K_t - \widetilde{K} = (K_0 - \widetilde{K})e^{\mu_1 t}. \tag{I.14}$$

Thus, one can rewrite (I.8) as

$$\dot{\tau}_t \;=\; -\frac{H^G_{\tau K}}{H^G_{\tau\tau}}\mu_1(K_0 - \widetilde{K})e^{\mu_1 t}. \tag{I.15}$$

Around a steady state, both $H^G_{\tau K}$ and $H^G_{\tau\tau}$ are constant and can be denoted as $\widetilde{H}^G_{\tau K}, \widetilde{H}^G_{\tau\tau}$. Thus, (I.15) can be solved by integration (using the terminal condition $\tau_\infty = \widetilde{\tau}$) to get

$$\tau_t \;=\; \widetilde{\tau} + \alpha(K_0 - \widetilde{K})e^{\mu_1 t}, \tag{I.16}$$

where

$$\alpha \;\equiv\; -\frac{\widetilde{H}^G_{\tau K}}{\widetilde{H}^G_{\tau\tau}}. \tag{I.17}$$

Note additionally that the impact of the capital stock in period t on the period t tax rate can be derived from (I.8), when one multiplies (I.8) by ∂t and solves for $\partial\tau_t/\partial K_t$:

$$\frac{\partial\tau_t}{\partial K_t} \;=\; -\frac{H^G_{\tau K}}{H^G_{\tau\tau}}. \tag{I.18}$$

Thus, α determines the slope of the function $\tau(K)$ around the steady state, i.e., $\alpha \equiv \partial\widetilde{\tau}/\partial K$. Due to $H^G_{\tau\tau} < 0$, the sign of α is determined by the sign of $H^G_{\tau K}$, which may be either positive or negative (see (I.10)). Hence, the optimal tax rate may either be an increasing or a decreasing function of the capital stock. □

J Proof of Proposition 5

Proposition 5 states that Proposition 1 holds in the case of a time-varying tax rate when one replaces τ^* by $\widetilde{\tau}$. The proof is straightforward. First, τ^* and $\widetilde{\tau}$ coincide (see Proposition 4). Moreover, $\widetilde{K}$ is determined by (10a) in both situations. Hence, Proposition 1 can be proven again using Equations (10a) and (18).

Second, Proposition 5 states that Proposition 2 is qualitatively unchanged. Because $\tau^* = \widetilde{\tau}$, all long term effects of a change in internal debt remain exactly the same as in Proposition 2. Moreover, if the economy is static ($\mu_1 \to -\infty$), the time-varying tax rate model collapses to a static model with a constant tax rate. Hence, Lemma 2 holds as well. Therefore, it remains to prove that Equation (21) holds. To derive the initial impact of b on welfare, differentiate Equation (F.12) with respect to b:

$$\frac{d\Omega_0}{db} \;=\; -\tau_0 r\widetilde{K} + \left[F(\widetilde{K},\widetilde{L}^m) - rb\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)(G(\widetilde{L}^d) - \widetilde{w}\widetilde{L}^d)\right]\frac{d\tau_0}{db}, \tag{J.1}$$

where τ_0 is the steady state tax rate $\widetilde{\tau}$ associated with the initial steady state capital stock $K_0 = \widetilde{K}$. Evaluate (J.1) at $\beta = 1$ and use the the constant returns property $F = F_K K + F_L L^m$:

$$\frac{d\Omega_0}{db}(\beta = 1) \quad = \quad -\widetilde{\tau} r \widetilde{K} + \left[F_K(\widetilde{K}, \widetilde{L}^m) - rb \right] \widetilde{K} \frac{d\tau_0}{db}(\beta = 1). \tag{J.2}$$

The change in the initial tax rate is, according to Equation (25):

$$\frac{d\tau_0}{db} \quad = \quad \frac{d\widetilde{\tau}}{db} - \alpha \frac{d\widetilde{K}}{db}. \tag{J.3}$$

We add and subtract $rC^{B\prime}\widetilde{K}$ from (J.2), and use (J.3) to simplify the resulting expression:

$$\begin{aligned}
\frac{d\Omega_0}{db}(\beta = 1) &= -\widetilde{\tau} r \widetilde{K} - rC^{B\prime}\widetilde{K} + rC^{B\prime}\widetilde{K} + (F_K - rb)\, \widetilde{K} \left[\frac{d\widetilde{\tau}}{db}(\beta = 1) - \alpha \frac{d\widetilde{K}}{db}(\beta = 1) \right] \\
&= \widetilde{K} \left[(F_K - rb) \left(\frac{r(C^{B\prime} - \widetilde{\tau})}{F_K - rb} + \frac{d\widetilde{\tau}}{db}(\beta = 1) - \alpha \frac{d\widetilde{K}}{db}(\beta = 1) \right) - rC^{B\prime} \right] \\
&= \widetilde{K} \left[\frac{(F_K - rb)\frac{d\widetilde{K}}{db}(\beta = 1)}{\frac{\partial \widetilde{K}}{\partial \tau}} \left(1 - \frac{\partial \widetilde{K}}{\partial \tau} \frac{\partial \widetilde{\tau}}{\partial K} \right) - rC^{B\prime} \right] < 0, \quad \text{if} \quad \frac{\partial \widetilde{K}}{\partial \tau} \frac{\partial \widetilde{\tau}}{\partial K} < 1, b = 0,
\end{aligned} \tag{J.4}$$

where I used (F.10) to derive the last row of (J.4) and replaced α by $\partial\widetilde{\tau}/\partial K$. Equation (10a) determines the steady state capital stock as a function of the tax rate (and, thus, $\partial\widetilde{K}/\partial\tau$), while (I.18) determines the optimal tax rate as a function of the capital stock (and, thus, $\partial\widetilde{\tau}/\partial K$). The steady state is stable if the product of the slopes of these functions is less than one, i.e., if $(\partial\widetilde{K}/\partial\tau)(\partial\widetilde{\tau}/\partial K) < 1$. Thus, Proposition 2 holds in the case of a time-varying tax rate when the steady state is stable.

The initial negative welfare impact, determined by (J.4), is more (less) pronounced than in the case of a constant tax rate if $\partial\widetilde{\tau}/\partial K$ is positive (negative).

Lastly, we derive the optimal TCR, similarly to Appendix G. The first-order condition with respect to b is given by (G.2), when one replaces τ by τ_t in the integrand. Hence, we can derive the optimal b^* analogously. The only difference is that we must replace (G.4) by the following equation, derived from (I.3)-(I.6):

$$\begin{aligned}
&\tau_t(F_K - rb) + \frac{\partial w_t}{\partial K_t}(1 - \tau_t) \left[L_t^m + (1 - \beta) L_t^d \right] \\
&= -\frac{r - I_t - \frac{\partial I_t}{\partial K_t} K_t}{\frac{\partial I_t}{\partial \tau_t} K_t} \left[F(K_t, L_t^m) - rbK_t - w_t L_t^m + (1 - \beta)[G(L_t^d - w_t L_t^d)] \right] - U_{KK} I_t K_t.
\end{aligned} \tag{J.5}$$

Inserting (J.5) in the first-order condition with respect to b, evaluating at $b = 0$, and then evaluating it around the steady state, gives the same result as (G.5). Thus, the optimal TCR is positive. Following the remaining steps from Appendix G, we can prove that that $0 < b^* < \hat{b}$ and that (23) holds. $\square$

K Model with endogenous domestic capital

Denote the domestic firm's value function, evaluated at period 0 as V_0^D. It can be derived similarly to the MNE's V_0 in Appendix A. Following the same steps, it is straightforward to show that

$$V_0^D = \int_0^\infty \pi_t^D e^{-rt} dt, \tag{K.1}$$

where π_t^D is defined in Equation (26). The domestic firm chooses L_t^d and I_t^d to maximize (K.1) subject to the equation of motion (27). Following the same steps as in Appendix B, the optimal labor demand L_t^d is again determined by Equation (2), while investment I_t^d follows

$$\dot{I}_t^d = \frac{1}{C^{d\prime\prime}} \left[r + C^d(I_t^d) + C^{d\prime}(I_t^d)(r - I_t^d) - G_K(K_t^d, L_t^d)(1-\tau) \right]. \tag{K.2}$$

Next, Equations (2), (8), and (9) determine the labor inputs L_t^d, L_t^m and wage rate w_t as implicit functions of the two capital stocks, K_t, K_t^d. That is, we have $L_t^m(K_t, K_t^d)$, $L_t^d(K_t, K_t^d), w_t(K_t, K_t^d)$. The partial derivatives with respect to K_t are again given by (11) in the main text. The partial derivatives with respect to K_t^d are derived analogously to (11) and are given by

$$\frac{\partial L_t^m}{\partial K_t^d} = \frac{G_{LK}}{F_{LL} + G_{LL}} < 0, \quad \frac{\partial L_t^d}{\partial K_t^d} = -\frac{\partial L_t^m}{\partial K_t}, \quad \frac{\partial w_t}{\partial K_t^d} = \frac{F_{LL} G_{LK}}{F_{LL} + G_{LL}} > 0. \tag{K.3}$$

Next, we derive the impact of changes in the tax rate τ and the TCR b on the two capital stocks, labor demands, and wage rate (analogously to Lemma 1). First, let us derive the steady state effects. The steady state capital stock $\widetilde{K}$ is determined by (10a), while $\widetilde{K}^d$ is detremined by (28) (i.e., the solution to (K.2) in steady state). Moreover, the steady state labor demands and wage rate are determined by (10b) and (10c). Solve (10c) for $\widetilde{L}^d$ and insert the solution in $G_L(\widetilde{K}^d, \widetilde{L}^d) = \widetilde{w}$. Totally differentiating (10a), the two equations in (10b), and (28) with respect to $\widetilde{K}, \widetilde{L}^m, \widetilde{K}^d, \widetilde{w}, \tau$, and b we get

$$\underbrace{\begin{pmatrix} F_{KK} & F_{KL} & 0 & 0 \\ F_{LK} & F_{LL} & 0 & -1 \\ 0 & -G_{KL} & G_{KK} & 0 \\ 0 & -G_{LL} & G_{LK} & -1 \end{pmatrix}}_{=J_2} \begin{pmatrix} \partial \widetilde{K} \\ \partial \widetilde{L}^m \\ \partial \widetilde{K}^d \\ \partial \widetilde{w} \end{pmatrix} = \begin{pmatrix} \frac{F_K - rb}{1-\tau} & \frac{r(C^{B\prime} - \tau)}{1-\tau} \\ 0 & 0 \\ \frac{G_K}{1-\tau} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial \tau \\ \partial b \end{pmatrix}. \tag{K.4}$$

The determinant of J_2 is

$$|J_2| = -\left[G_{KK}(F_{KK}F_{LL} - F_{KL}^2) + F_{KK}(G_{KK}G_{LL} - G_{KL}^2)\right] > 0, \tag{K.5}$$

where the positive sign is due to the assumption that at least one production function is characterized by decreasing returns to scale. Applying Cramer's rule to (K.4), we get

$$\frac{\partial \widetilde{K}}{\partial \tau} = \frac{-(F_K - rb)[G_{KK}(F_{LL} + G_{LL}) - G_{KL}^2] + G_K F_{KL} G_{KL}}{(1-\tau)|J_2|}, \tag{K.6}$$

$$\frac{\partial \widetilde{L}^m}{\partial \tau} = \frac{(F_K - rb)F_{KL}G_{KK} - G_K F_{KK} G_{KL}}{(1-\tau)|J_2|}, \tag{K.7}$$

$$\frac{\partial \widetilde{K}^d}{\partial \tau} = \frac{(F_K - rb)F_{KL}G_{KL} - G_K[F_{KK}(F_{LL} + G_{LL}) - F_{KL}^2]}{(1-\tau)|J_2|}, \tag{K.8}$$

$$\frac{\partial \widetilde{w}}{\partial \tau} = -\frac{(F_K - rb)F_{KL}[G_{KK}G_{LL} - G_{KL}^2] + G_K G_{KL}[F_{KK}F_{LL} - F_{KL}^2]}{(1-\tau)|J_2|} < 0, \tag{K.9}$$

$$\frac{\partial \widetilde{K}}{\partial b} = \frac{-r(C^{B\prime} - \tau)[G_{KK}(F_{LL} + G_{LL}) - G_{KL}^2]}{(1-\tau)|J_2|} > 0, \quad \text{if } b < \hat{b}, \tag{K.10}$$

$$\frac{\partial \widetilde{L}^m}{\partial b} = \frac{r(C^{B\prime} - \tau)F_{KL}G_{KK}}{(1-\tau)|J_2|} > 0, \quad \text{if } b < \hat{b}, \tag{K.11}$$

$$\frac{\partial \widetilde{K}^d}{\partial b} = \frac{r(C^{B\prime} - \tau)F_{KL}G_{KL}}{(1-\tau)|J_2|} < 0, \quad \text{if } b < \hat{b}, \tag{K.12}$$

$$\frac{\partial \widetilde{w}}{\partial b} = \frac{-r(C^{B\prime} - \tau)F_{KL}[G_{KK}G_{LL} - G_{KL}^2]}{(1-\tau)|J_2|} > 0, \quad \text{if } b < \hat{b}. \tag{K.13}$$

One important result is that a relaxation of the TCR lowers the domestic capital stock in the long term. The reason is that it stimulates investment by the MNE, which raises the MNE's labor demand and, hence, the wage rate. At a higher wage, the domestic firm lowers its own labor demand and therefore its optimal capital stock also declines.

Next, we derive the transitional dynamics of the capital stocks, following a change in one of the policy parameters. Consider first a change in the tax rate τ. The transition is determined by four differential equations: (4), (7), (27), and (K.2). I follow Wildasin (2011) in solving the system of differential equations. First, define the following change of variables:

$$y_1 \equiv \frac{\partial K_t}{\partial \tau}, \quad y_2 \equiv \frac{\partial K_t^d}{\partial \tau}, \tag{K.14}$$

$$x_1 \equiv \frac{\partial \dot{K}_t}{\partial \tau}, \quad x_2 \equiv \frac{\partial \dot{K}_t^d}{\partial \tau}. \tag{K.15}$$

Then, $x_i = \dot{y}_i$ for $i = 1, 2$. Now, differentiate (4), (7), (27), and (K.2) with respect to $K_t, \dot{K}_t, I_t, \dot{I}_t, K_t^d, \dot{K}_t^d, I_t^d, \dot{I}_t^d$, and τ. Following the same steps as in Equations (C.2)-(C.6) in Appendix C, and using the definitions from (K.14)-(K.15), we get

$$\dot{x}_1 - r x_1 - a_{11} y_1 - a_{12} y_2 = \Gamma_1, \tag{K.16}$$

$$\dot{x}_2 - r x_2 - a_{21} y_1 - a_{22} y_2 = \Gamma_2, \tag{K.17}$$

where

$$a_{11} \equiv -\frac{(1-\tau)\widetilde{K}}{C''} \frac{F_{KK}(F_{LL} + G_{LL}) - F_{KL}^2}{F_{LL} + G_{LL}}, \tag{K.18}$$

$$a_{12} \equiv -\frac{(1-\tau)\widetilde{K}}{C''} \frac{F_{KL} G_{KL}}{F_{LL} + G_{LL}}, \tag{K.19}$$

$$a_{21} \equiv -\frac{(1-\tau)\widetilde{K}^d}{C^{d\prime\prime}} \frac{F_{KL} G_{KL}}{F_{LL} + G_{LL}}, \tag{K.20}$$

$$a_{22} \equiv -\frac{(1-\tau)\widetilde{K}^d}{C^{d\prime\prime}} \frac{G_{KK}(F_{LL} + G_{LL}) - G_{KL}^2}{F_{LL} + G_{LL}}, \tag{K.21}$$

$$\Gamma_1 \equiv \frac{(F_K - rb)\widetilde{K}}{C''}, \tag{K.22}$$

$$\Gamma_2 \equiv \frac{G_K \widetilde{K}^d}{C^{d\prime\prime}}. \tag{K.23}$$

The system of four differential equations (K.16),(K.17), $\dot{y}_1 = x_1$, and $\dot{y}_2 = x_2$ is qualitatively identical to the system solved by Wildasin (2011) (presented by Equation (A.10) in his paper). The solution is also identical. The homogeneous part has two positive and two negative roots (see Wildasin (2011), Appendix A for a proof). The two negative roots are given by

$$\zeta_1 = \frac{r - \sqrt{b_1 + 2\sqrt{b_2}}}{2}, \quad \zeta_2 = \frac{r - \sqrt{b_1 - 2\sqrt{b_2}}}{2}, \tag{K.24}$$

where

$$b_1 = r^2 + 2(a_{11} + a_{22}), \tag{K.25}$$

$$b_2 = (a_{22} - a_{11})^2 + 4 a_{12} a_{21}. \tag{K.26}$$

The stable general solutions for $y_1 \equiv \frac{\partial K_t}{\partial \tau}$ and $y_2 \equiv \frac{\partial K_t^d}{\partial \tau}$ are given by (Turnovsky, 1997, p. 259):

$$\frac{\partial K_t}{\partial \tau} = \frac{\partial \widetilde{K}}{\partial \tau} + B_{1\tau} e^{\zeta_1 t} + B_{2\tau} e^{\zeta_2 t}, \tag{K.27}$$

$$\frac{\partial K_t^d}{\partial \tau} = \frac{\partial \widetilde{K}^d}{\partial \tau} + B_{1\tau} \psi_{21} e^{\zeta_1 t} + B_{2\tau} \psi_{22} e^{\zeta_2 t}, \tag{K.28}$$

where $B_{1\tau}, B_{2\tau}$ are constants to be determined by the initial conditions and ψ_{2i} are determined by the homogeneous solution as follows:

$$\begin{pmatrix} -\zeta_i & 0 & 1 & 0 \\ 0 & -\zeta_i & 0 & 1 \\ a_{11} & a_{12} & r - \zeta_i & 0 \\ a_{21} & a_{22} & 0 & r - \zeta_i \end{pmatrix} \begin{pmatrix} 1 \\ \psi_{2i} \\ \psi_{3i} \\ \psi_{4i} \end{pmatrix} = 0. \tag{K.29}$$

Solving (K.29) for ψ_{2i}, we get

$$\psi_{2i} = -\frac{(r - \zeta_i)\zeta_i + a_{11}}{a_{12}}, \quad i = 1, 2. \tag{K.30}$$

Moreover, the constants $B_{1\tau}$ and $B_{2\tau}$ are determined by the initial conditions $\partial K_0 / \partial \tau = 0$ and $\partial K_0^d / \partial \tau = 0$. Together (K.27),(K.28), and the initial conditions determine

$$B_{1\tau} = \frac{-\frac{\partial \widetilde{K}^d}{\partial \tau} + \psi_{22} \frac{\partial \widetilde{K}}{\partial \tau}}{\psi_{21} - \psi_{22}}, \tag{K.31}$$

$$B_{2\tau} = \frac{\frac{\partial \widetilde{K}^d}{\partial \tau} - \psi_{21} \frac{\partial \widetilde{K}}{\partial \tau}}{\psi_{21} - \psi_{22}}. \tag{K.32}$$

Furthermore, the impact of the TCR b on K_t, K_t^d can be derived analogously to the effect of τ.

Next, I determine the optimal tax τ^* and TCR b^*. The maximization problem is analogous to (G.1) in Appendix G. The first-order condition with respect to τ is, after some cancellation of terms including the changes in labor demands,

$$\begin{aligned} \frac{\partial}{\partial \tau} = \int_0^\infty \Bigg\{ & F(K_t, L_t^m) - rbK_t - w_t L_t^m + (1 - \beta)[G(K_t^d, L_t^d) - w_t L_t^d] \\ & + \left[\tau(F_K - rb) + (1 - \tau)[L_t^m + (1 - \beta)L_t^d] \frac{\partial w_t}{\partial K_t} \right] \frac{\partial K_t}{\partial \tau} \\ & + \left[\tau G_K + (1 - \tau)[L_t^m + (1 - \beta)L_t^d] \frac{\partial w_t}{\partial K_t^d} \right] \frac{\partial K_t^d}{\partial \tau} \end{aligned}$$

$$- \beta \left[\left(I_t^d + C^d(I_t^d) - (1-\tau) G_K \right) \frac{\partial K_t^d}{\partial \tau} + (1 + C^{d\prime}) K_t^d \frac{\partial I_t^d}{\partial \tau} \right] \Bigg\} e^{-rt} dt = 0. \quad \text{(K.33)}$$

Next, we evaluate (K.33) around steady state. In steady state, $I_t^d \approx \widetilde{I}^d = 0$ and thus $C^d(I_t^d) \approx 0$. Also, from (27), evaluated around steady state, we have $K_t^d \partial I_t^d / \partial \tau \approx \partial \dot{K}_t^d / \partial \tau$. Furthermore, from the domestic firm's maximization problem, around steady state, we get $1 + C^{d\prime} \approx (1-\tau) G_K / r$. Thus, (K.33) becomes

$$\begin{aligned} \frac{\partial}{\partial \tau} = \int_0^\infty \Bigg\{ & F(\widetilde{K}, \widetilde{L}^m) - rb\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)[G(\widetilde{K}^d, \widetilde{L}^d) - \widetilde{w}\widetilde{L}^d] \\ & + \left[\tau^*(F_K - rb) + (1-\tau)[\widetilde{L}^m + (1-\beta)\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \widetilde{K}} \right] \frac{\partial K_t}{\partial \tau} \\ & + \left[\tau G_K + (1-\tau)[\widetilde{L}^m + (1-\beta)\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \widetilde{K}^d} \right] \frac{\partial K_t^d}{\partial \tau} \\ & + \beta(1-\tau) G_K \left[\frac{\partial K_t^d}{\partial \tau} - \frac{1}{r} \frac{\partial \dot{K}_t^d}{\partial \tau} \right] \Bigg\} e^{-rt} dt = 0. \end{aligned} \quad \text{(K.34)}$$

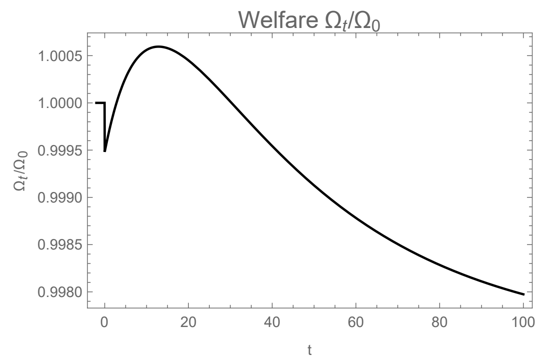
Integration of the last row of (K.34) by parts shows that it equals zero. Thus, only the first three rows of (K.34) remain. Next, it is straightforward to use (K.27) and (K.28) to integrate (K.34). Simplifying the resulting equation by expressing L^m as $1 - L^d$, we get

$$\begin{aligned} 0 = & F(\widetilde{K}, \widetilde{L}^m) - rb\widetilde{K} - \widetilde{w}\widetilde{L}^m + (1-\beta)[G(\widetilde{K}^d, \widetilde{L}^d) - \widetilde{w}\widetilde{L}^d] \\ & + \left[\tau^*(F_K - rb) + (1-\tau^*)[1 - \beta\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \widetilde{K}} \right] \left(\frac{\partial \widetilde{K}}{\partial \tau} + \frac{rB_{1\tau}}{r - \zeta_1} + \frac{rB_{2\tau}}{r - \zeta_2} \right) \\ & + \left[\tau^* G_K + (1-\tau^*)[1 - \beta\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \widetilde{K}^d} \right] \left(\frac{\partial \widetilde{K}^d}{\partial \tau} + \frac{rB_{1\tau}\psi_{21}}{r - \zeta_1} + \frac{rB_{2\tau}\psi_{22}}{r - \zeta_2} \right). \end{aligned} \quad \text{(K.35)}$$

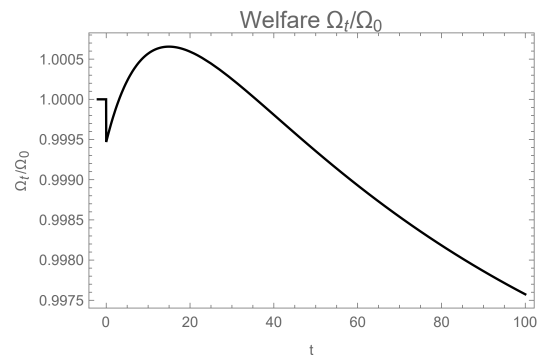
Analogously, the optimal TCR b^* is determined by

$$\begin{aligned} 0 = & -\tau^* r\widetilde{K} + \left[\tau^*(F_K - rb^*) + (1-\tau^*)[1 - \beta\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \widetilde{K}} \right] \left(\frac{\partial \widetilde{K}}{\partial b} + \frac{rB_{1b}}{r - \zeta_1} + \frac{rB_{2b}}{r - \zeta_2} \right) \\ & + \left[\tau G_K + (1-\tau^*)[1 - \beta\widetilde{L}^d] \frac{\partial \widetilde{w}}{\partial \widetilde{K}^d} \right] \left(\frac{\partial \widetilde{K}^d}{\partial b} + \frac{rB_{1b}\psi_{21}}{r - \zeta_1} + \frac{rB_{2b}\psi_{22}}{r - \zeta_2} \right). \end{aligned} \quad \text{(K.36)}$$

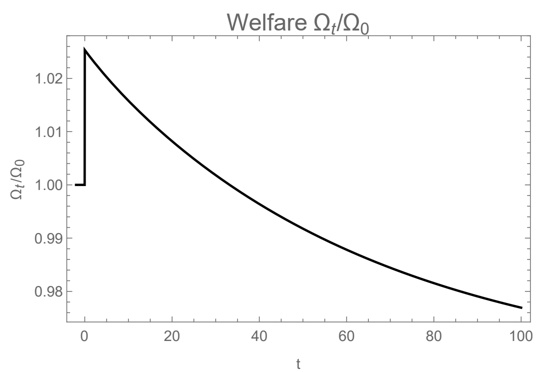
Simulation. Figure (K.1) presents the time path of welfare Ω_t relative to welfare before reform, Ω_0, for the four cases considered in the simulation in Section 5.2.



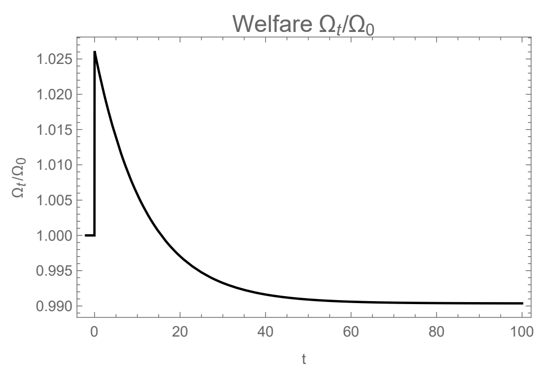
(K.1-A)



(K.1-B)



(K.1-C)



(K.1-D)

Figure K.1: Welfare effects for a change from $b = 0$ to $b = b^* = 0.279$ at time $t = 0$ for $\epsilon = 1, \nu = 0.6$ (panel (K.1-A)), $\epsilon = 1, \nu = 0.75$ (panel (K.1-B)), $\epsilon = 0.95, \nu = 0.95$ (panel (K.1-C)), and $\epsilon = 0.95, \nu = 0.85$ (panel (K.1-D)).

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